

ON HOMOLOGY OF COMPLEMENTS OF COMPACT SETS IN HILBERT CUBE

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ABSTRACT. We introduce the notion of spaces with weak relative cohomology and show the acyclicity of the complement $Q \setminus X$ in the Hilbert cube Q of a compactum X with weak relative cohomology. As a corollary we obtain the acyclicity of the complement results when

- (a) X is weakly infinite dimensional;
- (b) X has finite cohomological dimension.

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1. INTRODUCTION

This paper was motivated in part by a recent results of Belegardek and Hu [BH] about connectedness properties of the space $\mathcal{R}_{\geq 0}^k(\mathbb{R}^2)$ of non-negative curvature metrics on \mathbb{R}^2 where k stands for C^k topology. In particular, they proved that the complement $\mathcal{R}_{\geq 0}^k(\mathbb{R}^2) \setminus X$ is connected for every finite dimensional X . In [A] this result was extended to weakly infinite dimensional spaces X . The main topological idea of these results is that a weakly infinite dimensional compact set cannot separate the Hilbert cube.

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Banach and Zarichnyi posted in [P] a problem (Problem Q1053) whether the complement $Q \setminus X$ to a weakly infinite dimensional compactum X in the Hilbert cube Q is acyclic. They were motivated by the facts that $Q \setminus X$ is acyclic in the case when X is finite dimensional [K] and when it is countably dimensional (see [BCK]). It turns out that the problem was already answered affirmatively by Garity and Wright in the 80s [GW]. It follows from Theorem 4.5 [GW] which states that finite codimension closed subsets of the Hilbert cube are strongly infinite dimensional. In this paper we found some sufficient cohomological conditions on compact metric spaces for the acyclicity of the complement in the Hilbert cube which give an alternative solution for the Banach-Zarichnyi problem.

We introduced a cohomological version of the concept of strongly infinite dimensional spaces. We call a spaces which is not cohomologically strongly infinite dimensional as *spaces with weak relative cohomology*. The main result of the paper is an acyclicity statement for the complement of compacta with weak relative cohomology in the Hilbert cube.

2. MAIN THEOREM

2.1. Alexander Duality. We recall that for a compact set $X \subset S^n$ there is a natural isomorphism $AD_n : H^{n-k-1}(X) \rightarrow H_k(S^n \setminus X)$ called the Alexander duality. The naturality means that for a closed subset $Y \subset X$ there is a commutative diagram (1):

$$\begin{array}{ccc} H^{n-k-1}(X) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus X) \\ \downarrow & & \downarrow \\ H^{n-k-1}(Y) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus Y). \end{array}$$

Here we use the singular homology groups and the Čech cohomology groups. We note the Alexander Duality commutes with the suspension isomorphism s in the diagram (2):

$$\begin{array}{ccc} H^{n-k-1}(X) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus X) \\ s \downarrow & & \cong \downarrow \\ H^{n-k}(\Sigma X) & \xrightarrow{AD_{n+1}} & \tilde{H}_k(S^{n+1} \setminus \Sigma X). \end{array}$$

This the diagram can be viewed as a special case of the Spanier-Whitehead duality [W].

Since $B^n / \partial B^n = S^n$, the Alexander Duality in the n -sphere S^n and its property can be restated verbatim for the n -ball B^n for subspaces

$X \subset B^n$ with $X \cap \partial B^n \neq \emptyset$ in terms of relative cohomology groups

$$H^{n-k-1}(X, X \cap \partial B^n) \xrightarrow{AD} H_k(\text{Int } B^n \setminus X).$$

Note that for a pointed space X the reduced suspension ΣX is the quotient space $(X \times I)/(X \times \partial I \cup x_0 \times I)$ and the suspension isomorphism $s : H^i(X, x_0) \rightarrow H^{i+1}(\Sigma X) = H^{i+1}(X \times I, X \times \partial I \cup x_0 \times I)$ is defined by a cross product with the fundamental class $\phi \in H^1(I, \partial I)$, $s(\alpha) = \alpha \times \phi$. Note that $\alpha \times \phi = \alpha^* \cup \phi^*$ in $H^{i+1}(X \times I, X \times \partial I \cup x_0 \times I)$ where $\alpha^* \in H^i(X \times I, x_0 \times I)$ is the image of α under the induced homomorphism for the projection $X \times I \rightarrow X$ and $\phi^* \in H^1(X \times I, X \times \partial I)$ is the image of ϕ under the induced homomorphism defined by the projection $X \times I \rightarrow I$. Thus in view of an isomorphism $H_*(\text{Int } B^n \setminus X) \rightarrow H_*(B^n \setminus X)$, the commutative diagram (2) stated for B^n turns into the following (2')

$$\begin{array}{ccc} H^{n-k-1}(X, \partial X) & \xrightarrow{AD_n} & \tilde{H}_k(B^n \setminus X) \\ -\cup \phi^* \downarrow & & \cong \downarrow \\ H^{n-k}(X \times I, \partial X \times I \cup X \times \partial I) & \xrightarrow{AD_{n+1}} & \tilde{H}_k(B^{n+1} \setminus (X \times I)) \end{array}$$

where $\partial X = X \cap \partial I^n$.

2.2. Strongly infinite dimensional spaces. First, we recall some classical definitions which are due to Alexandroff [AP]. A map $f : X \rightarrow I^n$ to the n -dimensional cube is called *essential* if it cannot be deformed to ∂I^n through the maps of pairs $(I^n, f^{-1}(\partial I^n)) \rightarrow (I^n, \partial I^n)$. A map $f : X \rightarrow Q = \prod_{i=1}^{\infty} I$ to the Hilbert cube is called *essential* if the composition $p_n \circ f : X \rightarrow I^n$ is essential for every projection $p_n : Q \rightarrow I^n$ onto the factor. A compact space X is called *strongly infinite dimensional* if it admits an essential map onto the Hilbert cube. An infinite dimensional space which is not strongly infinite dimensional is called *weakly infinite dimensional*.

We call a relative cohomology class $\alpha \in H^k(X, A)$ *pure relative* if $j^*(\alpha) = 0$ for the inclusion of pairs homomorphism $j^* : H^k(X, A) \rightarrow H^k(X)$. This means that $\alpha = \delta(\beta)$ for $\beta \in \tilde{H}^{k-1}(A)$. In the case when $k = 1$ the cohomology class β can be represented by a map $f' : A \rightarrow S^0 = \partial I$. Let $f : X \rightarrow I$ be a continuous extension of f' . Then $\alpha = f^*(\phi)$ for the map $f : (X, A) \rightarrow (I, \partial I)$ of pairs where ϕ is the fundamental class.

2.2.1. Proposition. *Suppose that for a compact metric space X there a sequence of pure relative classes $\alpha_i \in H^1(X, A_i)$, $i = 1, 2, \dots$, $A_i \subset_{cl} X$ such that the cup-product $\alpha_1 \cup \alpha_n \cup \dots \cup \alpha_n \neq 0$ for all n . Then X is strongly infinite dimensional.*

Proof. Let $f_i : (X, A_i) \rightarrow (I, \partial I)$ be representing α_i maps. Note that the homomorphism $f_n^* : H^n(I^n, \partial I^n) \rightarrow H^n(X, \cup_{i=1}^n A_i)$ induced by the map $\tilde{f}_n = (f_1, f_2, \dots, f_n) : X \rightarrow I^n$ takes the product of the fundamental classes $\phi_1 \cup \dots \cup \phi_n$ to $\alpha_1 \cup \alpha_n \cup \dots \cup \alpha_n \neq 0$. Therefore, f_n is essential. Thus, the map

$$f = (f_1, f_2, \dots) : X \rightarrow \prod_{i=1}^{\infty} I = Q$$

is essential. \square

We call a space X *cohomologically strongly infinite dimensional* if there is a sequence of pure relative classes $\alpha_i \in H^{k_i}(X, A_i)$, $k_i > 0$, $i = 1, 2, \dots$, $A_i \subset_{cl} X$ such that the cup-product $\alpha_1 \cup \alpha_n \cup \dots \cup \alpha_n \neq 0$ for all n . Compacta which are not cohomologically strongly infinite dimensional will be called as *compacta with weak relative cohomology* or *relative cohomology weak compacta*. One can take any coefficient ring R to define relative cohomology weak compacta with respect to R .

Thus, the class of cohomologically weak compacta contains weakly infinite dimensional compacta (Proposition 3.0.6) as well as all compacta with finite cohomological dimension.

2.3. Proof of the main theorem.

2.3.1. Proposition. *Let $X \subset Q$ be a closed subset and let $a \in H_i(Q \setminus X)$ be a nonzero element for some i . Then there exists an $n \in \mathbb{N}$ such that the image a_n of a under the inclusion homomorphism*

$$H_i(Q \setminus X) \rightarrow H_i(Q \setminus p_n^{-1}p_n(X))$$

is nontrivial.

Proof. This follows from the fact that

$$H_i(Q \setminus X) = \varinjlim H_i(Q \setminus p_n^{-1}p_n(X)).$$

\square

2.3.2. Theorem. *Let $X \subset Q$ be a compact subset of the Hilbert cube with weak relative cohomology with coefficients in a ring with unit R . Then $\tilde{H}_i(Q \setminus X; R) = 0$ for all i .*

Proof. We assume everywhere below that the coefficients are taken in R .

We note that $Q \setminus p_n^{-1}p_n(X) = (I^n \setminus p_n(X)) \times Q_n$ where Q_n is the Hilbert cube complementary to I^n in $Q = I^n \times Q_n$.

Assume the contrary, $\tilde{H}_i(Q \setminus X) \neq 0$. Let $a \in \tilde{H}_i(Q \setminus X)$ be a nonzero element. We use the notation $X_k = p_k(X)$. By Proposition 2.3.1 there is n such that $a_k \neq 0$, $a_k \in \tilde{H}_i((I^k \setminus X_k) \times Q_k)$ for all $k \geq n$. Thus, we may assume that a_k lives in $\tilde{H}_i((I^k \setminus X_k) \times \{0\}) \cong \tilde{H}_i((I^k \setminus X_k) \times Q_k)$.

Let $\beta_k \in H^{k-i-1}(X_k, X_k \cap \partial I^k)$ be dual to a_k for the Alexander duality applied in $(I^k, \partial I^k)$. By the suspension isomorphism for the Alexander duality (see diagram (2')), a_k is dual in I^{k+1} to $\beta_k^* \cup \phi^*$ where β_k^* is the image of β_k under the homomorphism induced by the projection $X_k \times I \rightarrow X_k$ and ϕ^* is the image of the fundamental class $\phi \in H^1(I, \partial I)$ under the homomorphism induced by the projection $X_k \times I \rightarrow I$.

By the naturality of the Alexander duality (we use the B^n -version of the diagram (1)) applied to the inclusion $X_{k+1} \subset X_k \times I$ we obtain that a_{k+1} is dual to $\beta'_{k+1} \cup \phi_{k+1}$ where β'_{k+1} is the restriction of β_k^* to X_{k+1} and ϕ_{k+1} is the restriction of ϕ^* to X_{k+1} . Note that $\beta'_{k+1} = (q_k^{k+1})^*(\beta_k)$ where $q_k^{k+1} : X_{k+1} \rightarrow X_k$ is the projection $p_k^{k+1} : I^{k+1} \rightarrow I^k$ restricted to X_{k+1} . Thus, $\beta_{k+1} = (q_k^{k+1})^*(\beta_k) \cup \phi_{k+1}$.

By induction on m we obtain

$$(*) \quad \beta_{n+m} = (q_n^{n+m})^*(\beta_n) \cup \psi_1 \cup \dots \cup \psi_m$$

where $\psi_j = (q_{n+j}^{n+m})^*(\phi_{n+j})$, $j = 1, \dots, m$.

We define a sequence $\alpha_j = (q_{n+j})^*(\phi_{n+j})$ of relative cohomology classes on X where $q_n = p_n|_X : X \rightarrow X_n$. Since $X = \varprojlim X_r$, for the Čech cohomology we have

$$H^\ell(X, q_{n+k}^{-1}(A)) = \varinjlim \{H^\ell(X_{n+m}, (q_{n+k}^{n+m})^{-1}(A))\}_{m \geq k}$$

for any $A \subset_{Cl} X_{n+k}$. Hence for fixed k the product $\alpha_1 \cup \dots \cup \alpha_k$ is defined by the thread

$$\{(q_{n+1}^{n+m})^*(\phi_{n+1}) \cup \dots \cup (q_{n+k}^{n+m})^*(\phi_{n+k})\}_{m \geq k}$$

in the above direct limit. Since $\beta_{n+m} \neq 0$, in view of $(*)$ we obtain that

$$(q_{n+1}^{n+m})^*(\phi_{n+1}) \cup \dots \cup (q_{n+k}^{n+m})^*(\phi_{n+k}) = \psi_1 \cup \dots \cup \psi_k \neq 0.$$

Hence, $\alpha_1 \cup \dots \cup \alpha_k \neq 0$ for all k . Therefore, X is cohomologically strongly infinite dimensional. This contradicts to the assumption. \square

2.3.3. Corollary. *Let $X \subset Q$ be a compactum with finite cohomological dimension. Then $\tilde{H}_i(Q \setminus X) = 0$ for all i .*

We recall that the cohomological dimension of a space X with integers as the coefficients is defined as

$$\dim_{\mathbb{Z}} X = \sup\{n \mid \exists A \subset_{Cl} X \text{ with } H^n(X, A) \neq 0\}.$$

2.3.4. Corollary. *Let $X \subset Q$ be weakly infinite dimensional compactum. Then $\tilde{H}_i(Q \setminus X) = 0$ for all i .*

Proof. This is rather a corollary of the proof of Theorem 2.3.2 where we constructed a nontrivial infinite product of pure relative 1-dimensional cohomology classes $\alpha_1 \cup \alpha_2 \cup \dots$ on X . Then the corollary holds true in view of Proposition 2.2.1. \square

3. ON WEAKLY INFINITE DIMENSIONAL COMPACTA

Let $\phi \in H^{m-1}(K(G, m-1); G)$ be the fundamental class. Let $\bar{\phi}$ denote the image $\delta(\phi) \in H^m(\text{Cone}(K(G, m-1)), K(G, m-1); G)$ of the fundamental class under the connecting homomorphism in the exact sequence of pair.

3.0.5. Proposition. *For any pure relative cohomology class $\alpha \in H^m(X, A; G)$ there is a map of pairs*

$$f : (X, A) \rightarrow (\text{Cone}(K(G, m-1)), K(G, m-1))$$

such that $\alpha = f^(\bar{\phi})$.*

Proof. From the exact sequence of the pair (X, A) it follows that $\alpha = \delta(\beta)$ for some $\beta \in H^{m-1}(A; G)$. Let $g : A \rightarrow K(G, m-1)$ be a map representing β . Since $\text{Cone}(K(G, m-1))$ is an AR, there is an extension $f : X \rightarrow \text{Cone}(K(G, m-1))$. Then the commutativity of the diagram formed by f^* and the exact sequence of pairs implies $\delta g^*(\phi) = f^*\delta(\phi) = f^*(\bar{\phi})$. \square

Corollary 2.3.4 can be derived formally from the following:

3.0.6. Theorem. *Every weakly infinite dimensional compactum is relative cohomology weak for any coefficient ring R .*

Proof. Let $\alpha_i \in H^{k_i}(X, A_i; R)$ be a sequence of pure relative classes with a nontrivial infinite cup-product. We may assume that $k_i > 1$. We denote $K_i = K(R, k_i)$. By Proposition 3.0.5 there are maps $f_i : (X, A_i) \rightarrow (\text{Cone}(K_i), K_i)$ representing α_i in a sense that $f_i^*(\bar{\phi}_i) = \alpha_i$. Let

$$q_i = \text{Cone}(c) : \text{Cone}(K_i) \rightarrow \text{Cone}(pt) = [0, 1]$$

be the cone over the constant map $c : K_i \rightarrow \{0\}$. We show that $g = (q_1 f_1, q_2 f_2, \dots) : X \rightarrow \prod_{i=1}^{\infty} I$ is essential. Assume that for some n the map

$$g_n = (q_1 f_1, q_2 f_2, \dots, q_n f_n) : X \rightarrow \prod_{i=1}^n I = I^n$$

is inessential. Then there is a deformation H of g_n to a map $h : X \rightarrow N_\epsilon(\partial I^n)$ rel $g_n^{-1}(N_\epsilon(\partial I^n))$ to the ϵ -neighborhood of the boundary ∂I^n . Since the map $q_1 \times \cdots \times q_n$ is a fibration over $\text{Int } I^n$, there is a lift \bar{H} of H that deforms

$$\bar{f}_n = (f_1, \dots, f_n) : X \rightarrow \prod_{i=1}^n \text{Cone}(K_i)$$

to an ϵ -neighborhood of $M = (q_1 \times \cdots \times q_n)^{-1}(\partial I^n)$. Since M is an ANR we may assume that \bar{f}_n can be deformed to M rel $\bar{f}_n^{-1}(M)$. Since $\alpha_1 \cup \cdots \cup \alpha_n = \bar{f}_n^*(\bar{\phi}_1 \cup \cdots \cup \bar{\phi}_n)$, to complete the proof it suffices to show that the restriction of $\bar{\phi}_1 \cup \cdots \cup \bar{\phi}_n$ to M is zero.

We denote

$$D = \prod_{i=1}^n \text{Cone}(K_i) \setminus \prod_{i=1}^n \text{Ocone}(K_i)$$

where $\text{Ocone}(K)$ stands for the open cone. Since $\prod \text{Cone}(K_i)$ is contractible, $\phi_1 \cup \cdots \cup \phi_n = \delta\omega$ for some $\omega \in H^{k_1+\cdots+k_n-1}(D; R)$.

We show that D is homeomorphic to $K_1 * \cdots * K_n$. Note that

$$\prod_{i=1}^n \text{Cone}(K_i) = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1]\}$$

and

$$\prod_{i=1}^n \text{Ocone}(K_i) = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1)\}$$

with the convention that $0x = 0x'$. Then the set D consists of all $t_1x_1 + \cdots + t_nx_n$ such that $t_i = 1$ for some i . We recall that

$$K_1 * \cdots * K_n = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1] \sum t_i = 1\}$$

with the same convention. Clearly, the projection of the unit sphere in \mathbb{R}^n for the $\max\{|x_i|\}$ norm onto the unit sphere for the $|x_1| + \cdots + |x_n|$ norm is bijective. Thus, the renormalizing map $\rho : D \rightarrow K_1 * \cdots * K_n$,

$$\rho(t_1x_1 + \cdots + t_nx_n) = \frac{t_1}{\sum t_i}x_1 + \cdots + \frac{t_n}{\sum t_i}x_n$$

is bijective and, hence, is a homeomorphism.

The space M can be defined as follows:

$$M = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1] \exists i : t_i = 0\}.$$

Therefore, $M = D \cup L$ where

$$L = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1] \exists i : t_i = 1\}.$$

We note that L is the cone over

$$L^* = \{t_1x_1 + \cdots + t_nx_n \in K_1 * \cdots * K_n \mid \exists i : t_i = 0\}.$$

Thus, the space M is obtained by attaching to D the cone over L^* .

We show that the inclusion $L^* \rightarrow D$ is null-homotopic. Note that

$$L^* = \bigcup_{i=1}^n K_1 * \cdots * \hat{K}_i * \cdots * K_n$$

where \hat{K}_i means that the i -factor is missing. Fix points $e_i \in K_i$ and the straight-line deformation

$$h_t^i : K_1 * \cdots * \hat{K}_i * \cdots * K_n \rightarrow K_1 * \cdots * K_n$$

to the point e_i . We recall that the reduced joint product is defined as follows

$$K_1 \tilde{*} \cdots \tilde{*} K_n = \{t_1x_1 + \cdots + t_nx_n \mid x_i \in K_i, t_i \in [0, 1], \sum t_i = 1\}$$

with convention that $0x = 0x'$ and $e_i = e_j$. Let $q : K_1 * \cdots * K_n \rightarrow K_1 \tilde{*} \cdots \tilde{*} K_n$ be the quotient map. Note that q has contractible fibers and hence is a homotopy equivalence. Then $\tilde{x}_0 = q(\Delta^{n-1})$ is the base point in the reduced joint product where the simplex Δ^{n-1} is spanned by e_1, \dots, e_n .

The deformation h_t^i defines a deformation \tilde{h}_t^i of $K_1 \tilde{*} \cdots \tilde{*} \hat{K}_i \tilde{*} \cdots \tilde{*} K_n$ to the base point $\sum t_i e_i$ in the reduced joint product $K_1 \tilde{*} \cdots \tilde{*} K_n$. We note that for any $i < j$ the deformations \tilde{h}_t^i and \tilde{h}_t^j agree on the common part $K_1 \tilde{*} \cdots \tilde{*} \hat{K}_i \tilde{*} \cdots \tilde{*} \hat{K}_j \tilde{*} \cdots \tilde{*} K_n$. Therefore, the union $(\cup \tilde{h}_t^i) \circ q|_{L^*}$ is a well-defined deformation of L^* to the base point in the reduced joint product where $q : K_1 * \cdots * K_n \rightarrow K_1 \tilde{*} \cdots \tilde{*} K_n$ is quotient map. Since q is a homotopy equivalence, L^* is null-homotopic in $K_1 * \cdots * K_n$. \square

Since there exist strongly infinite dimensional compacta with finite cohomological dimension (see [D],[DW]) the converse to Theorem 3.0.6 does not hold true.

The following can be easily derived from the definition of the cup product for the singular cohomology and the definition of the Čech cohomology.

3.0.7. Proposition. *Let $A, A' \subset Y$, $Y, B, B' \subset X$ be closed subsets with $B \cap Y = A$ and $B' \cap Y = A'$. Then for any ring R there is a*

commutative diagram generated by inclusions and the cup product

$$\begin{array}{ccc} H^k(X, B; R) \times H^l(X, B'; R) & \xrightarrow{\cup} & H^{k+l}(X, B \cup B'; R) \\ \downarrow & & \downarrow \\ H^k(Y, A; R) \times H^l(Y, A'; R) & \xrightarrow{\cup} & H^{k+l}(Y, A \cup A'; R). \end{array}$$

3.0.8. Theorem. *Let Y be a closed subset of a compactum X with a weak relative cohomology over a ring R . Then Y has a weak relative cohomology over R .*

Proof. Assume that Y is cohomologically strongly infinite dimensional. Let $\alpha_i \in H^{k_i}(Y, A_i; R)$ be pure relative cohomology classes with nonzero product. Let

$$f_i : (Y, A_i) \rightarrow (\text{Cone}(K(R, k_i - 1)), K(R, k_i - 1))$$

be a map from Proposition 3.0.5 representing α_i . Let

$$g_i : X \rightarrow \text{Cone}(K(R, k_i - 1))$$

be an extension of f_i and let $B_i = g_i^{-1}(K(R, k_i - 1))$. We consider $\beta_i = g_i^*(\bar{\phi}_i) \in H^{k_i}(X, B_i)$. Note that $\alpha_i = \xi_i^*(\beta_i)$ where $\xi_i : (Y, A_i) \rightarrow (X, B_i)$ is the inclusion. By induction, Proposition 3.0.7, and the fact $\alpha_i \cup \dots \cup \alpha_n \neq 0$ we obtain that $\beta_1 \cup \dots \cup \beta_n \neq 0$ for all n . Then we can conclude that X is cohomologically strongly infinite dimensional. \square

4. APPLICATIONS

Let Σ denote the linear span of the standard Hilbert cube Q in the Hilbert space ℓ_2 and let Σ^ω denote the product of countably many copies of Σ . The following is well-known:

4.0.9. Proposition. *Let H be either ℓ_2 or Σ^ω . Then for any closed subset $A \subset Q$ of the Hilbert cube any embedding $\phi : A \rightarrow H$ can be extended to an embedding $\bar{\phi} : Q \rightarrow H$.*

Let $\mathcal{R}^\gamma(M)$ denote the space of all Riemannian metrics on a manifold M with the C^γ topology. For the definition of such topology for finite γ which are not integers we refer to [BB]. The following theorem was proved in [BB]:

4.0.10. Theorem. *For a connected manifold M the space $\mathcal{R}^\gamma(M)$ is homeomorphic to Σ^ω when $\gamma < \infty$ and $\mathcal{R}^\infty(M)$ is homeomorphic to ℓ_2 .*

The space $\mathcal{R}_{\geq 0}^\gamma(\mathbb{R}^2)$ is homeomorphic to Σ^ω when $\gamma < \infty$ and it is not an integer. The space $\mathcal{R}_{\geq 0}^\infty(\mathbb{R}^2)$ is homeomorphic to ℓ_2 .

This theorem with our main result implies the following

4.0.11. Corollary. *Let $X \subset \mathcal{R}^\gamma(M)$ be a closed weakly infinite dimensional subset for a connected manifold M . Then $\mathcal{R}^\gamma(M) \setminus X$ is acyclic.*

Let $X \subset \mathcal{R}_{\geq 0}^\gamma(\mathbb{R}^2)$ be a closed weakly infinite dimensional subset when $\gamma = \infty$ or $\gamma < \infty$ and it is not an integer. Then the complement $\mathcal{R}_{\geq 0}^\gamma(\mathbb{R}^2) \setminus X$ is acyclic.

Proof. In either of the cases we denote the space of metrics by H . Let $f : K \rightarrow H \setminus X$ be a singular cycle. By Proposition 4.0.9 there is an embedding of the Hilbert cube $Q \subset H$ such that $f(K) \subset Q$. By Theorem 2.3.2 f is homologous to zero in $Q \setminus X$ and, hence, in $H \setminus X$. \square

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